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Some New Hilbert-Type Fractional Inequalities

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Abstract
The main purpose of this article is to present some new Hilbert-type fractional integral inequalities by using katugampola Fractional Calculus. The strict inequality is considered and the best possible constant is determined for general and some other special cases.

Keywords:
Katugampola fractional derivative; Katugampola fractional integral; Hilbert's inequality.

1. Introduction:

In recent years, the fractional calculus and its applications has received a great interest by researchers. As a consequence, this affected positively on the field of integral inequalities since they have been widely used in the science and engineering.

Katugampola Fractional Calculus is a new kind of Fractional Calculus introduced and studied by Katugampola in the year 2014. It is more suited to the approximation of classical calculus. In [8], the author gave the definitions of Katugampola fractional derivative and Katugampola fractional integral. We will give the necessary notation and basic definitions below. For more details one may refer to [8].

Definition 1.1

Let $f:[0,\infty)\to R$ and $t>0$. Then the "Katugampola fractional derivative" of f of order α is defined by

$$D^{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t\,e^{\varepsilon t^{-\alpha}})-f(t)}{\varepsilon}, \qquad (1.\,1)$$

for $t>0$ and $\alpha\in(0,1]$. If f is $\alpha-$differentiable in some open interval $(0,a)$, $a>0$ and $\lim_{t\to 0^+}D^{\alpha}(f)(t)$ exists, then

$$D^{\alpha}(f)(0)=\lim_{t\to 0^+}D^{\alpha}(f)(t).$$

If f is differentiable, then we have

$$D^{\alpha}(f)(t)=t^{1-\alpha}\frac{d\,f(t)}{dt}.$$

Definition 1.2 (Katugampola Fractional integral)

Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in R$. Then, the α – fractional integral of f is defined by,

$$I_a^{\alpha}(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\, dx \qquad (1.2)$$

if the Riemann improper integral exists.

Definition 1.3 [2]

Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a, b] \to R$ is α – fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(t)\, d_\alpha t := \int_a^b f(t)\, t^{\alpha-1} dt \qquad (1.3)$$

exists.

Many authors have given considerable attention to the following well-known integral Hilbert's inequality which is important in analysis and its applications [3].

If f, g are nonnegative real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$, and $0 < \int_0^\infty g^2(y)dy < \infty$, then

we have [4]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\, dxdy < \pi\left(\int_0^\infty f^2(x)dx\right)^{1/2}\left(\int_0^\infty g^2(y)dy\right)^{1/2}, \qquad (1.4)$$

where π is the best possible constant .

Inequality (1. 4) has been studied and generalized in many directions by a number of distinguished mathematicians(see [3, 5, 6]). They expended considerable effort in finding best possible constants for the new inequalities they deduced.

Li et al. [10] have proved the following Hardy- Hilbert's type inequality using the hypotheses of (1. 4):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}\, dx\, dy < c\left(\int_0^\infty f^2(x)\, dx\right)^{1/2}\left(\int_0^\infty g^2(y)\, dy\right)^{1/2}, \qquad (1.5)$$

where the constant factor $c = \sqrt{2}\left(\pi - 2 - \tan^{-1}\sqrt{2}\right) = 1.7408\ldots$ is the best possible .

Qiong liu and Wenbing Sun [7] have studied the following Hilbert-type integral inequality:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}\, dx\, dy < 4\left(\int_0^\infty f^2(x)\, dx\right)^{1/2}\left(\int_0^\infty g^2(y)\, dy\right)^{1/2}, \qquad (1.6)$$

and then they employed the local fractional calculus (see [9] for notations and full definitions) to deduce a new Hilbert-type fractional integral inequality given in the theorem below :

Theorem 1.1

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ 0 < \alpha \leq 1,\ f, g(>0) \in C_\alpha(0,\infty),$

and $0 < {}_0I_\infty^\alpha(x^{\frac{\alpha}{2}(p-2)} f^p(x)) < \infty,$

$0 < {}_0I_\infty^\alpha(y^{\frac{\alpha}{2}(q-2)} g^q(y)) < \infty,$ *then*

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$$ {}_0I_\infty^\alpha\left[{}_0I_\infty^\alpha \frac{f(x)g(y)}{\max\{x^\alpha,y^\alpha\}}\right]< $$

$$ \eta(\alpha)\left\{{}_0I_\infty^\alpha\left(x^{\frac{\alpha}{2}(p-2)}f^p(x)\right)\right\}^{\frac{1}{p}}\left\{{}_0I_\infty^\alpha\left(y^{\frac{\alpha}{2}(q-2)}g^q(y)\right)\right\}^{\frac{1}{q}} $$

where the constant factor $\eta(\alpha)=\dfrac{2^{\alpha+1}}{\Gamma(1+\alpha)}$ is the best possible.

2. Main Results

In article [1], the author obtained some new extensions of Hilbert's type inequality and gave some important applications. The main result was the inequality

$$ \int_0^\infty\int_0^\infty \frac{|\ln x-\ln y|^c}{a\,x+b\,y+|x-y|}f(x)g(y)\,dx\,dy $$

$$ <A\left(\int_0^\infty f^2(x)\,dx\right)^{1/2}\left(\int_0^\infty g^2(y)\,dy\right)^{1/2} $$

where a, b, and c are three non-negative real numbers and the constant A is given by

$$ A=\int_0^1 \frac{2^{c+1}|\ln(t)|^c}{(a+1)+t^2(b-1)}\,dt+\int_0^1 \frac{2^{c+1}|\ln(t)|^c}{(b+1)+t^2(a-1)}\,dt $$

In this paper, we introduce some new fractional analogs of Hilbert's type inequalities that obtained in [1] by using katugampola Fractional Calculus. We need the following lemma to establish our main results.

Lemma 2.1 Let c, a, and b be three non-negative real numbers and $\alpha\in(0,1]$. Then we have the following equations

$$ \int_0^\infty \frac{\alpha^{c+1}|\ln x-\ln y|^c}{a\,x^\alpha+b\,y^\alpha+|x^\alpha-y^\alpha|}\left(\frac{x}{y}\right)^{\alpha/2}d_\alpha y= $$

$$ \int_0^\infty \frac{\alpha^{c+1}|\ln x-\ln y|^c}{a\,x^\alpha+b\,y^\alpha+|x^\alpha-y^\alpha|}\left(\frac{y}{x}\right)^{\alpha/2}d_\alpha x $$

$$ =\int_0^{\alpha^{\frac{1}{\alpha}}} \frac{2^{c+1}\alpha^2|\alpha\ln(t)-\ln\alpha|^c}{\alpha^2(a+1)+t^{2\alpha}(b-1)}d_\alpha t+ $$

$$ \int_0^{\alpha^{\frac{1}{\alpha}}} \frac{2^{c+1}\alpha^2|\alpha\ln(t)-\ln\alpha|^c}{\alpha^2(b+1)+t^{2\alpha}(a-1)}d_\alpha t=A $$

where $A:=A(c,a,b)\in[0,\infty]$.

Proof. For any given $y>0$, let $y^\alpha=\dfrac{(x\,t)^\alpha}{\alpha}$, then it follows that

$$ I=\int_0^\infty \frac{\alpha^{c+1}|\ln x-\ln y|^c}{a\,x^\alpha+b\,y^\alpha+|x^\alpha-y^\alpha|}\left(\frac{x}{y}\right)^{\alpha/2}d_\alpha y= $$

$$ \int_0^\infty \frac{\alpha|\alpha\ln(t)-\ln\alpha|^c}{a\,\alpha+b\,t^\alpha+|\alpha-t^\alpha|}\left(\frac{\alpha}{t^\alpha}\right)^{1/2}d_\alpha t $$

$$ =\int_0^{\alpha^{\frac{1}{\alpha}}} \frac{\alpha\,|\alpha\ln(t)-\ln\alpha|^c}{\alpha(a+1)+t^\alpha(b-1)}\left(\frac{\alpha}{t^\alpha}\right)^{1/2}d_\alpha t+ $$

$$ \int_{\alpha^{\frac{1}{\alpha}}}^\infty \frac{\alpha\,|\alpha\ln(t)-\ln\alpha|^c}{\alpha(a-1)+t^\alpha(b+1)}\left(\frac{\alpha}{t^\alpha}\right)^{1/2}d_\alpha t. $$

For the last integral, we let $t^\alpha=\alpha^2 s^{-\alpha}$ and rewrite this integral in terms of t^α, we then have

$$I=\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{\alpha\left|\alpha\ln(t)-\ln\alpha\right|^c}{\alpha(a+1)+t^\alpha(b-1)}\left(\frac{\alpha}{t^\alpha}\right)^{\frac{1}{2}}d_\alpha t+\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{\alpha\left|\alpha\ln(t)-\ln\alpha\right|^c}{t^\alpha(a-1)+\alpha(b+1)}\left(\frac{\alpha}{t^\alpha}\right)^{\frac{1}{2}}d_\alpha t$$

Upon setting $h^\alpha=\left(\alpha t^\alpha\right)^{\frac{1}{2}}$, we get

$$I=\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{2^{c+1}\alpha^2\left|\alpha\ln(h)-\ln\alpha\right|^c}{\alpha^2(a+1)+h^{2\alpha}(b-1)}d_\alpha h+\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{2^{c+1}\alpha^2\left|\alpha\ln(h)-\ln\alpha\right|^c}{h^{2\alpha}(a-1)+\alpha^2(b+1)}d_\alpha h=A$$

which implies the desired result.

Theorem 2.1 Let f and g be two $\alpha-$fractional integrable functions on $[0,\infty)$ such that

$$0<\int_0^\infty f^2(x)d_\alpha x<\infty,\ 0<\int_0^\infty g^2(y)d_\alpha y<\infty,$$

then we have

$$\int_0^\infty\int_0^\infty\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{a\,x^\alpha+b\,y^\alpha+\left|x^\alpha-y^\alpha\right|}f(x)g(y)d_\alpha x\,d_\alpha y$$

$$<A\left(\int_0^\infty f^2(x)d_\alpha x\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(y)d_\alpha y\right)^{\frac{1}{2}}\qquad(2.1)$$

where A is defined in lemma 2.1 and it is the best possible constant.

Proof.

By Cauchy-Schwartz inequality and Lemma 2.1, we get

$$\int_0^\infty\int_0^\infty\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{a\,x^\alpha+b\,y^\alpha+\left|x^\alpha-y^\alpha\right|}f(x)g(y)d_\alpha x\,d_\alpha y$$

$$=\int_0^\infty\int_0^\infty\left\{\left[\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{a\,x^\alpha+b\,y^\alpha+\left|x^\alpha-y^\alpha\right|}\right]^{\frac{1}{2}}\left(\frac{x^\alpha}{y^\alpha}\right)^{\frac{1}{4}}f(x)\right\}\times\left\{\left[\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{a\,x^\alpha+b\,y^\alpha+\left|x^\alpha-y^\alpha\right|}\right]^{\frac{1}{2}}\left(\frac{y^\alpha}{x^\alpha}\right)^{\frac{1}{4}}g(y)\right\}d_\alpha x\,d_\alpha y$$

$$\le\left\{\int_0^\infty\left(\int_0^\infty\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{a\,x^\alpha+b\,y^\alpha+\left|x^\alpha-y^\alpha\right|}\left(\frac{x^\alpha}{y^\alpha}\right)^{\frac{1}{2}}d_\alpha y\right)f^2(x)d_\alpha x\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_0^\infty\left(\int_0^\infty\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{a\,x^\alpha+by^\alpha+\left|x^\alpha-y^\alpha\right|}\left(\frac{y^\alpha}{x^\alpha}\right)^{\frac{1}{2}}d_\alpha x\right)f^2(x)d_\alpha y\right\}^{\frac{1}{2}}$$

$$\le A\left(\int_0^\infty f^2(x)d_\alpha x\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(y)d_\alpha y\right)^{\frac{1}{2}}.\qquad(2.2)$$

To accomplish the proof of Theorem 2.1, we will show that inequality (2.2) takes the form of strict inequality as well as that the constant A is best possible. We first show that the Inequality is strict:

If equation (2.2) takes the form of equality, then there exist constants d and e, such that they are not all zero and

$$d\,\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{ax^\alpha+by^\alpha+\left|x^\alpha-y^\alpha\right|}f^2(x)\left(\frac{x^\alpha}{y^\alpha}\right)^{\frac{1}{2}}=e\,\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{ax^\alpha+by^\alpha+\left|x^\alpha-y^\alpha\right|}g^2(y)\left(\frac{y^\alpha}{x^\alpha}\right)^{\frac{1}{2}},$$

a.e. on $(0,\infty)\times(0,\infty)$, then we have

$d\,x^{\alpha} f^{2}(x) = e\,y^{\alpha} g^{2}(y),$ a.e. on $(0,\infty)\times(0,\infty)$.

Hence

$$d\,x^{\alpha}\ f^{2}(x) = e\,y^{\alpha}\ g^{2}(y) = \text{ constant},$$

a.e. on $(0,\infty)\times(0,\infty)$.

Without loss of generality, suppose $d \neq 0$, then

$$f^{2}(x) = \frac{\text{constant}}{d} x^{-\alpha}, \quad \text{a.e. on } (0,\infty)\times(0,\infty).$$

Thus,

$$\int_{0}^{\infty} f^{2}(x) d_{\alpha} x = \frac{\text{constant}}{d} \int_{0}^{\infty} x^{-\alpha}\, d_{\alpha} x,$$

which contradicts the fact that $0 < \int_{0}^{\infty} f^{2}(x) d_{\alpha} x < \infty$. Hence (2. 2) takes the form of strict inequality, and thus we have (2.1).

Next, we show that A is the best possible constant of (2.1) . For $0 < \varepsilon < 1$, setting

$$f_{\varepsilon}(x) = \alpha^{(\varepsilon+1)/2}\ x^{-\alpha(\varepsilon+1)/2},$$

for $x \in [b,\infty]$; $f_{\varepsilon}(x) = 0$, for $x \in (a,b)$, (2.3)

also

$$g_{\varepsilon}(y) = \alpha^{\frac{(\varepsilon+1)}{2}}\ y^{-\alpha(\varepsilon+1)/2}, \text{ for}$$

$y \in [b,\infty]$; $g_{\varepsilon}(y) = 0$, for $y \in (a,\,b)$. (2.4)

Assume that the constant factor A in (2.1) is not the best possible, then there exists a positive number M with $M < A$ and $d > 0$; such that

$$\int_{d}^{\infty}\int_{0}^{\infty} \frac{\alpha^{c+1}\left|\ln x - \ln y\right|^{c}}{a\,x^{\alpha} + b\,y^{\alpha} + \left|x^{\alpha} - y^{\alpha}\right|} f(x) g(y) d_{\alpha} x\, d_{\alpha} y$$
$$< M\left(\int_{d}^{\infty} f^{2}(x)\, d_{\alpha} x\right)^{1/2} \left(\int_{d}^{\infty} g^{2}(y)\, d_{\alpha} y\right)^{1/2}.$$

Consequently,

$$\int_{d}^{\infty}\int_{0}^{\infty} \frac{\alpha^{c+1}\left|\ln x - \ln y\right|^{c}}{a\,x^{\alpha} + b\,y^{\alpha} + \left|x^{\alpha} - y^{\alpha}\right|} f_{\varepsilon}(x) g_{\varepsilon}(y) d_{\alpha} x\, d_{\alpha} y$$
$$< M\left(\int_{d}^{\infty} f_{\varepsilon}^{2}(x) d_{\alpha} x\right)^{1/2} \left(\int_{d}^{\infty} g_{\varepsilon}^{2}(y) d_{\alpha} y\right)^{1/2}$$

$$= M\left(\int_{d}^{\infty} \alpha^{\varepsilon+1} x^{-\alpha(\varepsilon+1)}\, d_{\alpha} x\right)^{1/2} \left(\int_{d}^{\infty} \alpha^{\varepsilon+1} y^{-\alpha(\varepsilon+1)}\, d_{\alpha} y\right)^{1/2}$$

$$= M\,\alpha^{\varepsilon} \big/ \varepsilon d^{\alpha\varepsilon}\ . \tag{2.5}$$

By (2.3) and (2.4) and setting $y^{\alpha} = \dfrac{(xu)^{\alpha}}{\alpha}$, we have

$$\int_{d}^{\infty}\int_{0}^{\infty} \frac{\alpha^{c+1}\left|\ln x - \ln y\right|^{c}}{a\,x^{\alpha} + b\,y^{\alpha} + \left|x^{\alpha} - y^{\alpha}\right|} f_{\varepsilon}(x) g_{\varepsilon}(y)\ d_{\alpha} x\, d_{\alpha} y$$

$$= \alpha^{\varepsilon+1} \int_{d}^{\infty}\int_{e}^{\infty} \frac{\alpha\left|\ln\left(\frac{x^{\alpha}}{y^{\alpha}}\right)\right|^{c}}{a\,x^{\alpha} + b\,y^{\alpha} + \left|x^{\alpha} - y^{\alpha}\right|}\, x^{\frac{-\alpha(\varepsilon+1)}{2}}\, y^{\frac{-\alpha(\varepsilon+1)}{2}}\ d_{\alpha} x\, d_{\alpha} y$$

$$=\alpha^{\varepsilon+1}\int_{d}^{\infty}\int_{\alpha e/x^\alpha}^{\infty}\frac{\alpha\left|\ln\left(\frac{\alpha}{u^\alpha}\right)\right|^c}{a\,x^\alpha+b\frac{x^\alpha u^\alpha}{\alpha}+\left|x^\alpha-\frac{x^\alpha u^\alpha}{\alpha}\right|}\times$$

$$x^{\frac{-\alpha(\varepsilon+1)}{2}}\left(\frac{(xu)^\alpha}{\alpha}\right)^{\frac{-(\varepsilon+1)}{2}}\frac{x^\alpha}{\alpha}\,d_\alpha x\,d_\alpha u$$

$$=(\alpha)^{\frac{3(\varepsilon+1)}{2}}\int_{d}^{\infty}\int_{\alpha e/x^\alpha}^{\infty}\frac{\alpha\left|\ln\alpha-\alpha\ln(u)\right|^c}{a\,\alpha+b\,u^\alpha+\left|\alpha-u^\alpha\right|}x^{-\alpha(\varepsilon+1)}u^{\frac{-\alpha(\varepsilon+1)}{2}}\,d_\alpha x\,d_\alpha u.$$

By (2.5) and for $e\to 0^+$, we have

$$(\alpha)^{\frac{3(\varepsilon+1)}{2}}\int_{d}^{\infty}\int_{0}^{\infty}\frac{\alpha\left|\ln\alpha-\alpha\ln(u)\right|^c}{a\,\alpha+b\,u^\alpha+\left|\alpha-u^\alpha\right|}\times$$

$$x^{-\alpha(\varepsilon+1)}u^{\frac{-\alpha(\varepsilon+1)}{2}}\,d_\alpha x\,d_\alpha u<M\alpha^\varepsilon/\varepsilon d^{\alpha\varepsilon},$$

that is

$$\frac{(\alpha)^{\frac{3(\varepsilon+1)}{2}}}{\alpha\,\varepsilon\,d^{\alpha\varepsilon}}\int_{0}^{\infty}\frac{\alpha\left|\ln\alpha-\alpha\ln(u)\right|^c}{a\,\alpha+b\,u^\alpha+\left|\alpha-u^\alpha\right|}u^{\frac{-\alpha(\varepsilon+1)}{2}}\,d_\alpha u<M\alpha^\varepsilon/\varepsilon d^{\alpha\varepsilon}.$$

We then have

$$\int_{0}^{\infty}\frac{\left|\ln\alpha-\alpha\ln(u)\right|^c}{a\,\alpha+b\,u^\alpha+\left|\alpha-u^\alpha\right|}u^{\frac{-\alpha(\varepsilon+1)}{2}}\,d_\alpha u<M/\alpha^{\frac{1+\varepsilon}{2}}.\quad(2.6)$$

When $e\to 0^+$, we get

$$\int_{0}^{\infty}\frac{\left|\ln\alpha-\alpha\ln(u)\right|^c}{a\,\alpha+b\,u^\alpha+\left|\alpha-u^\alpha\right|}u^{\frac{-\alpha(\varepsilon+1)}{2}}\,d_\alpha u$$

$$=\int_{0}^{\infty}\frac{\left|\ln\alpha-\alpha\ln(u)\right|^c}{a\,\alpha+b\,u^\alpha+\left|\alpha-u^\alpha\right|}u^{\frac{-\alpha}{2}}\,d_\alpha u+o(1)$$

$$=A+o(1).$$

Since, for $\varepsilon>0$ small enough, we have $A\le M$ which contradicts the hypothesis. Hence the constant factor A in (2.1) is the best possible. ■

3. Special Cases

We now show how one can establish several special inequalities by choosing different values for c, a, and b. Let us first introduce the following corollary :

Corollary 3.1 Under the hypotheses of Theorem 2.1, we obtain

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{\alpha^{c+1}\left|\ln x-\ln y\right|^c}{x^\alpha+y^\alpha+\left|x^\alpha-y^\alpha\right|}f(x)g(y)\,d_\alpha x\,d_\alpha y$$

$$<A_c\left(\int_{0}^{\infty}f^2(x)\,d_\alpha x\right)^{1/2}\left(\int_{0}^{\infty}g^2(y)\,d_\alpha y\right)^{1/2},\quad(3.1)$$

where $A=A_c$ is defined in lemma 2.1 with $a=b=1$, and it is the best possible.

Remark 3.1 For Inequality (3.1) which is a special case of (2.1), we obtain the following results :

(1) If $c=0$, then inequality (3. 1) reduces to

$$\int_0^\infty\int_0^\infty \frac{\alpha\, f(x)g(y)}{x^\alpha+y^\alpha+|x^\alpha-y^\alpha|}\,d_\alpha x\,d_\alpha y < A_0\left(\int_0^\infty f^2(x)\,d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)\,d_\alpha y\right)^{1/2},$$

where

$$A_0 = 2\int_0^{\alpha^{\frac{1}{\alpha}}} d_\alpha t = 2\int_0^{\alpha^{\frac{1}{\alpha}}} t^{\alpha-1}\,dt = 2.$$

(2) If $c=1$, then inequality (3.1) reduces to

$$\int_0^\infty\int_0^\infty \frac{\alpha^2|\ln x-\ln y|}{x^\alpha+y^\alpha+|x^\alpha-y^\alpha|}f(x)g(y)\,d_\alpha x\,d_\alpha y < A_1\left(\int_0^\infty f^2(x)\,d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)\,d_\alpha y\right)^{1/2},$$

and the constant

$$A=A_1=2\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{2^2\alpha^2|\alpha\ln(t)-\ln\alpha|}{2\alpha^2}d_\alpha t = -4\int_0^{\alpha^{\frac{1}{\alpha}}}\ln\left(t^\alpha/\alpha\right)d_\alpha t = 4$$

(3) If $c=2$, then we get from (3.1)

$$\int_0^\infty\int_0^\infty \frac{\alpha^3|\ln x-\ln y|^2}{x^\alpha+y^\alpha+|x^\alpha-y^\alpha|}f(x)g(y)\,d_\alpha x\,d_\alpha y < A_2\left(\int_0^\infty f^2(x)d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)d_\alpha y\right)^{1/2},$$

where $A_2 = 8\int_0^{\alpha^{\frac{1}{\alpha}}}\ln^2\left(\frac{t^\alpha}{\alpha}\right)d_\alpha t = 16.$

(4) If $c=3$, then we get from (3.1)

$$\int_0^\infty\int_0^\infty \frac{\alpha^4|\ln x-\ln y|^3}{x^\alpha+y^\alpha+|x^\alpha-y^\alpha|}f(x)g(y)\,d_\alpha x\,d_\alpha y < A_3\left(\int_0^\infty f^2(x)d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)d_\alpha y\right)^{1/2},$$

where

$$A_3 = 16\int_0^{\alpha^{\frac{1}{\alpha}}}\left|\ln\left(\frac{t^\alpha}{\alpha}\right)\right|^3 d_\alpha t = -16\int_0^{\alpha^{\frac{1}{\alpha}}}\ln^3\left(\frac{t^\alpha}{\alpha}\right)d_\alpha t = 96.$$

Remark 3.2 In the previous special inequalities, it is clear that we get the same results for the best constants that obtained by Abu Hany [1], and hence we reach to the same formula :

$$A_c = 2^{c+1}(c!),\quad for\ c=0,1,2,\ldots .$$

More special inequalities are obtained below from inequality (2.1), by choosing different values for c, a, and b as follows **:**

(1) If $c=a=0$, $b=1$, then we obtain

$$\int_0^\infty\int_0^\infty \frac{\alpha\, f(x)g(y)}{y^\alpha+|x^\alpha-y^\alpha|}\,d_\alpha x\,d_\alpha y < A\left(\int_0^\infty f^2(x)\,d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)\,d_\alpha y\right)^{1/2},$$

where

$$A=\int_0^{\alpha^{\frac{1}{\alpha}}} 2\,d_\alpha t+\int_0^{\alpha^{\frac{1}{\alpha}}} \frac{2}{2-\left(\frac{t^\alpha}{\alpha}\right)^2}\,d_\alpha t$$

$$=2+\frac{1}{\sqrt{2}}\int_0^{\alpha^{\frac{1}{\alpha}}}\left(\frac{1}{\left(\sqrt{2}-\frac{t^\alpha}{\alpha}\right)}+\frac{1}{\left(\sqrt{2}-\frac{t^\alpha}{\alpha}\right)}\right)d_\alpha t$$

$$=2+\frac{1}{\sqrt{2}}\left(-\ln\left(\sqrt{2}-\frac{t^\alpha}{\alpha}\right)+\ln\left(\sqrt{2}+\frac{t^\alpha}{\alpha}\right)\right)\Bigg|_0^{\alpha^{\frac{1}{\alpha}}}$$

$$=2+\frac{1}{\sqrt{2}}\left(-\ln\left(\sqrt{2}-1\right)+\ln\left(\sqrt{2}+1\right)\right).$$

$$=3.24646$$

(2) If $c=0,\ a=1,\ b=2,$ then

$$\int_0^\infty\int_0^\infty \frac{\alpha}{x^\alpha+2y^\alpha+\left|x^\alpha-y^\alpha\right|} f(x)g(y)\,d_\alpha x\,d_\alpha y$$

$$<A\left(\int_0^\infty f^2(x)\,d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)\,d_\alpha y\right)^{1/2},$$

where

$$A=\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{2\alpha^2}{2\alpha^2+t^{2\alpha}}\,d_\alpha t+\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{2}{3}\,d_\alpha t$$

$$=\frac{2}{3}+\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{2}{2+\frac{t^{2\alpha}}{\alpha^2}}\,d_\alpha t$$

$$=\frac{2}{3}+\sqrt{2}\tan^{-1}\left(\frac{t^\alpha}{\sqrt{2}\alpha}\right)\Bigg|_0^{\alpha^{\frac{1}{\alpha}}}$$

$$=\frac{2}{3}+\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sqrt{2}\tan^{-1}(0)$$

$$=1.5371.$$

(3) If $c=1,\ a=b=0,$ then

$$\int_0^\infty\int_0^\infty \frac{\alpha^2\left|\ln x-\ln y\right|}{\left|x^\alpha-y^\alpha\right|} f(x)g(y)\,d_\alpha x\,d_\alpha y$$

$$<A\left(\int_0^\infty f^2(x)\,d_\alpha x\right)^{1/2}\left(\int_0^\infty g^2(y)\,d_\alpha y\right)^{1/2},$$

where

$$A=8\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{\alpha^2\left|\ln\left(\frac{t^\alpha}{\alpha}\right)\right|}{\alpha^2-t^{2\alpha}}\,d_\alpha t$$

$$=-8\int_0^{\alpha^{\frac{1}{\alpha}}}\frac{\alpha^2\ln\left(\frac{t^\alpha}{\alpha}\right)}{\alpha^2-t^{2\alpha}}\,d_\alpha t$$

$$=-8\left[\sum_{k=1}^\infty\frac{\left(1-\frac{t^\alpha}{\alpha}\right)^k}{k^2}+\sum_{k=1}^\infty\frac{\left(-\frac{t^\alpha}{\alpha}\right)^k}{k^2}\right]_0^{\alpha^{\frac{1}{\alpha}}}$$

$$-8\left[\ln\left(\frac{t^\alpha}{\alpha}\right)\ln\left(\frac{t^\alpha}{\alpha}+1\right)\right]_0^{\alpha^{\frac{1}{\alpha}}}$$

$$= 4\left[\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \right]$$

$$= 8 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$= \pi^2.$$

(4) If $c = a = 0,\ b = 2,$ then we have

$$\int_0^{\infty}\int_0^{\infty} \frac{\alpha\, f(x)g(y)}{2\,y^{\alpha} + |x^{\alpha} - y^{\alpha}|}\, d_{\alpha}x\, d_{\alpha}y$$

$$< A\left(\int_0^{\infty} f^2(x) d_{\alpha} x\right)^{1/2} \left(\int_0^{\infty} g^2(y) d_{\alpha} y\right)^{1/2},$$

Where

$$A = \int_0^{\alpha^{\frac{1}{\alpha}}} \frac{2\alpha^2}{\alpha^2 + t^{2\alpha}}\, d_{\alpha}t + \int_0^{\alpha^{\frac{1}{\alpha}}} \frac{2\alpha^2}{3\alpha^2 - t^{2\alpha}}\, d_{\alpha}t$$

$$= 2\tan^{-1}\left(\frac{t^{\alpha}}{\alpha}\right)\Bigg|_0^{\alpha^{\frac{1}{\alpha}}} + \frac{1}{\sqrt{3}}\left(\ln\left(\sqrt{3} + \frac{t^{\alpha}}{\alpha}\right) - \ln\left(\sqrt{3} - \frac{t^{\alpha}}{\alpha}\right)\right)\Bigg|_0^{\alpha^{\frac{1}{\alpha}}}$$

$$= \frac{\pi}{2} + \frac{1}{\sqrt{3}}\left(\ln\left(\sqrt{3}+1\right) - \ln\left(\sqrt{3}-1\right)\right)$$

$$= 2.33114.$$

Remark 3.3 In the previous special inequalities, if we take $\alpha = 1,$ then we get the results obtained by Abu Hany [1].

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